Quantum Langevin dynamics of a multiport, multimode superconducting circuit

APS March Meeting

DQI, Superconducting Circuit Modeling Session

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LLNL-PRES-XXXXXX

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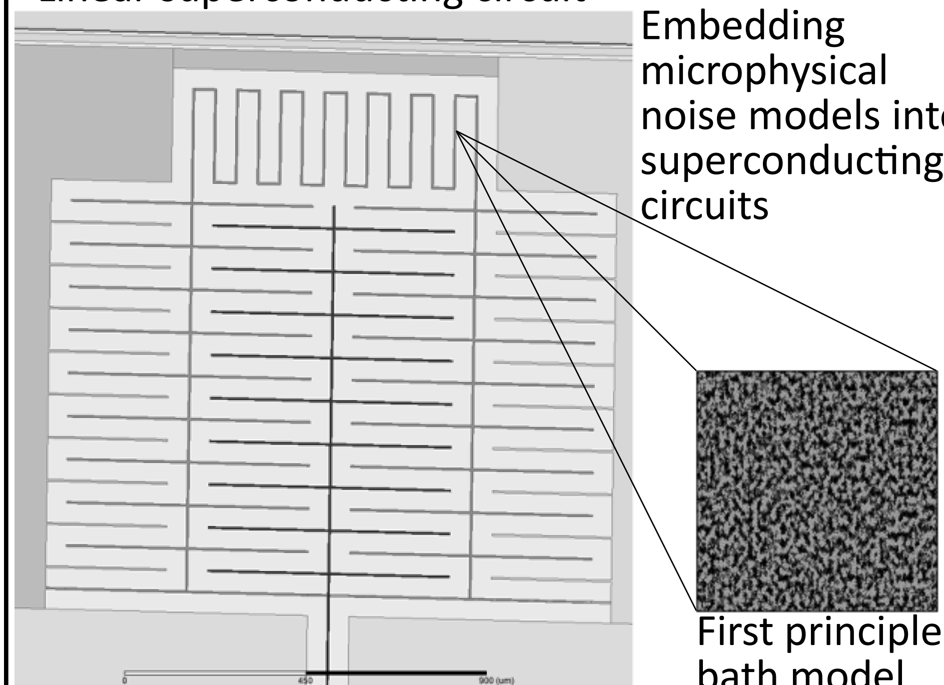
Problem Statement and Approach

Problem statement:

- Few dynamical models of superconducting circuits include noise processes derived from rigorous microphysical arguments

Proposed approach:

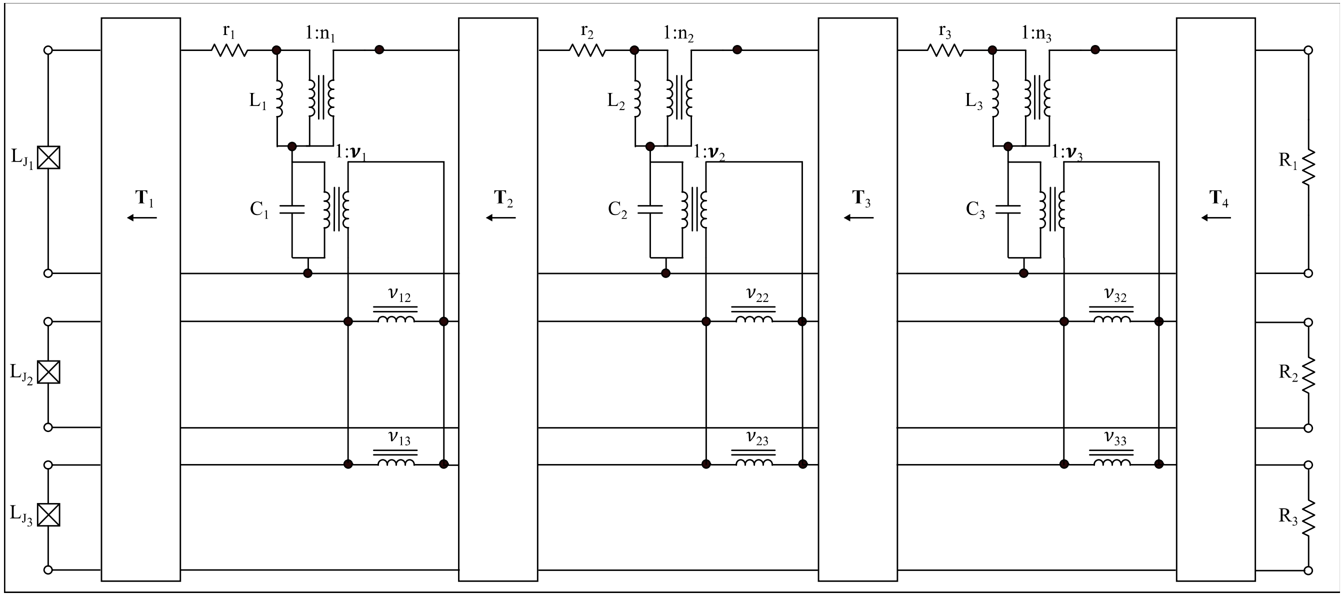
- To develop models using quantum Langevin equations to include general descriptions for the bath or multiple baths and their interaction with the system



FEM model of surface two level system device with ab initio bath – See Yaniv Rosen's talk, Friday 9:24 am, 501B and Keith Ray's talk, Friday 12:15 pm, 408B

Model system – 3 port, 3 stage Brune circuit

$$\hat{H}_S = \frac{1}{2}\sum_{ij}\left(\mathcal{C}_{0ij}^{-1}\hat{q}_i\hat{q}_j + M_{0ij}\hat{\phi}_i\hat{\phi}_j\right) - \left(\frac{\Phi_0}{2\pi}\right)^2 \sum_{j\leq N_{\mathrm{JJ}}} L_{J_j}^{-1}\cos\left(\frac{2\pi}{\Phi_0}\hat{\phi}_j\right)$$



Solgun et al. Ann. Phys. **361** (2015)
Solgun, PhD Thesis, RWTH, (2015)

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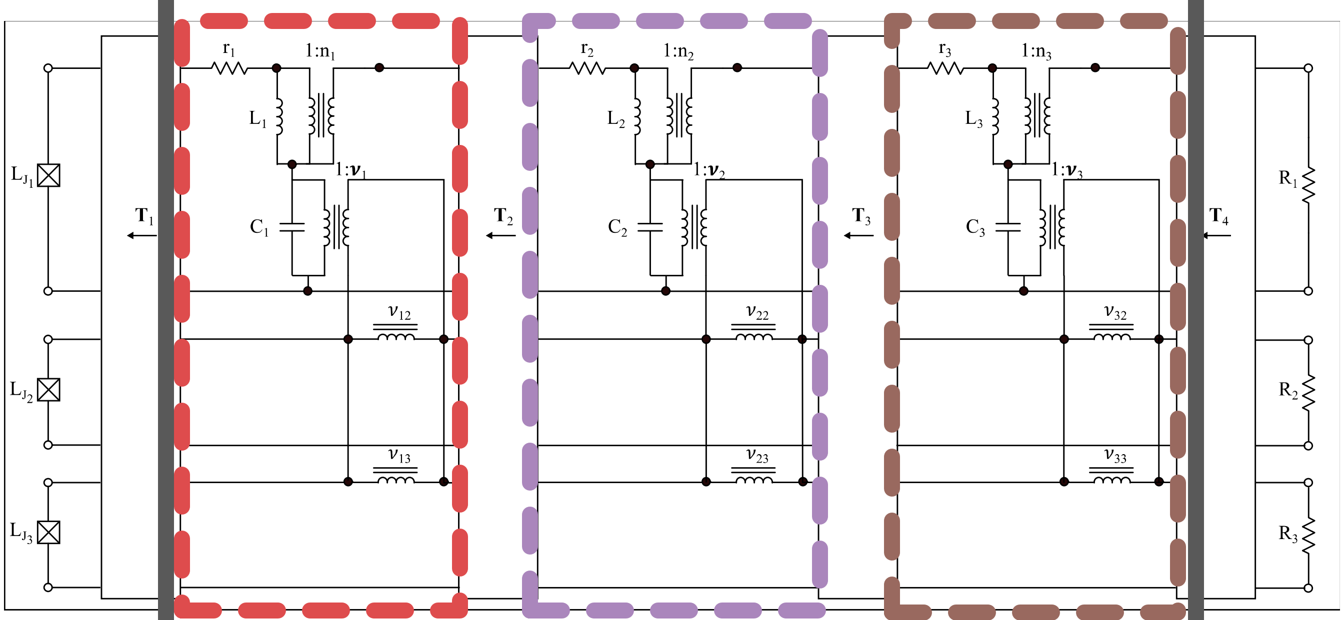
Linear Circuit Modes



- Linear part of the device is encapsulated in Brune circuit

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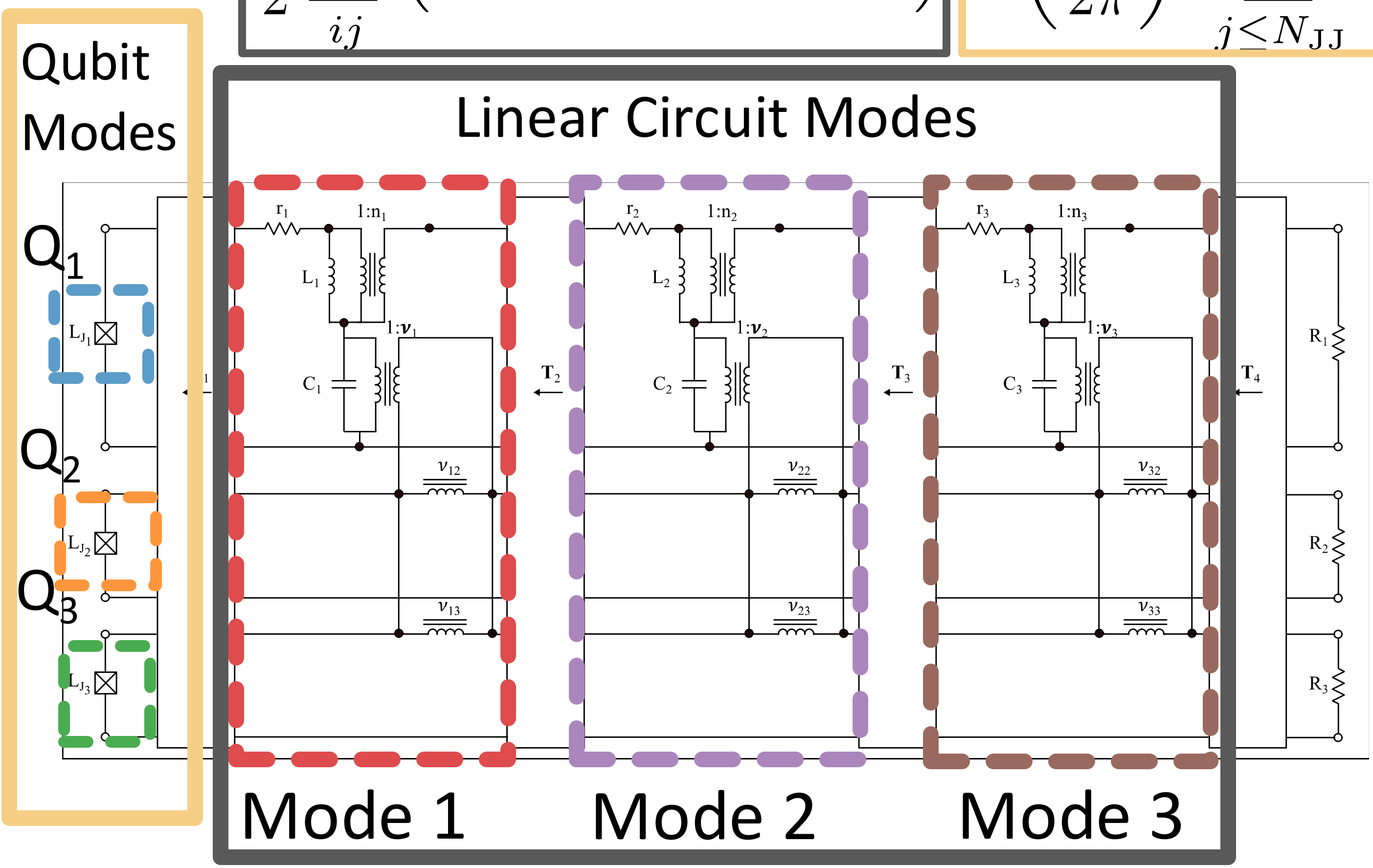

- Linear part of the device is encapsulated in Brune circuit
- All non-linear elements are applied to the ports

Solgun et al. Ann. Phys. **361** (2015)
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We derived the Langevin equations for a multiport Brune circuit Hamiltonian

Quantum Langevin equations

Charges: $$\frac{d\hat{q}_k}{dt} + \frac{i}{\hbar}\left[\hat{H}_S^{(m)},\ \hat{q}_k\right] + \lambda^2 \int_0^t \mathscr{K}\left(t-\tau\right)\dot{\phi}_k\left(\tau\right)d\tau = \lambda\hat{\mathscr{F}}(t)$$

Fluxes: $$\frac{d\hat{\phi}_k}{dt} = \frac{1}{2}\left(\sum_i \mathcal{C}_{0ik}^{-1}\hat{q}_i + \sum_j \mathcal{C}_{0kj}^{-1}\hat{q}_j\right)$$

Cortes et al. J.Chem.Phys. **82** (6) 1985

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Fluctuation (dephasing)

Dissipation (relaxation)

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Fluctuation (dephasing)

Dissipation (relaxation)

Fluctuation dissipation relation (FDR)

$$\mathscr{K}\left(t-\tau\right) = \sum_\alpha \Phi_\alpha(t-\tau)\frac{\tanh\left(\hbar w_\alpha/2k_BT\right)}{\hbar\omega_\alpha}$$

$$\Phi_\alpha(t-\tau) = \left\langle \hat{\mathscr{F}}(t)\hat{\mathscr{F}}(\tau) + \hat{\mathscr{F}}(\tau)\hat{\mathscr{F}}(t)\right\rangle$$

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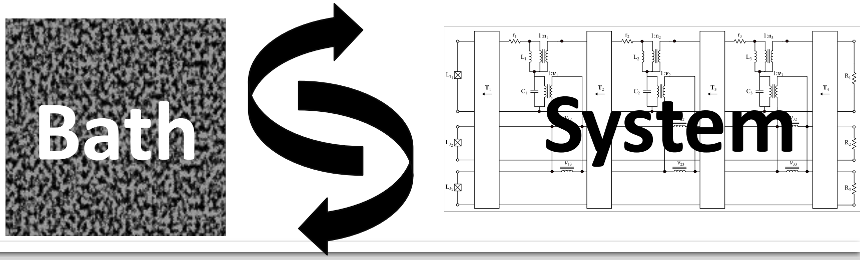
Bilinear system-bath coupling drives leads to additive noise form of Langevin equations and FDR



Mean equations of charge and flux

Mean equations

Charges:
$$\frac{d}{dt}\langle\hat{q}_k\rangle+\frac{1}{2}\left(\sum_i M_{0ik}\left\langle\hat{\phi}_i\right\rangle+\sum_j M_{0kj}\left\langle\hat{\phi}_j\right\rangle\right)-\frac{\Phi_0}{2\pi}\sum_{j\leq N_{\mathrm{JJ}}}L_{J_j}^{-1}\delta_{kj}\left\langle\sin\left(\frac{2\pi}{\Phi_0}\hat{\phi}_j\right)\right\rangle$$
$$-\lambda^2\sum_\alpha\frac{c_\alpha^2}{m_\alpha\omega_\alpha^2}\left\langle\hat{\phi}_k\right\rangle+\lambda^2\int_0^t\mathscr{K}(t-\tau)\left\langle\dot{\hat{\phi}}_k(\tau)\right\rangle d\tau=0$$

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Inductive coupling

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Josephson junction non-linearity

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Frequency shift

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Dissipative memory kernel

Cortes et al. J.Chem.Phys. **82** (6) 1985

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Markovian Limit

$$\mathscr{K}(t-\tau) \to \delta(t-\tau)$$

Cortes et al. J.Chem.Phys. **82** (6) 1985

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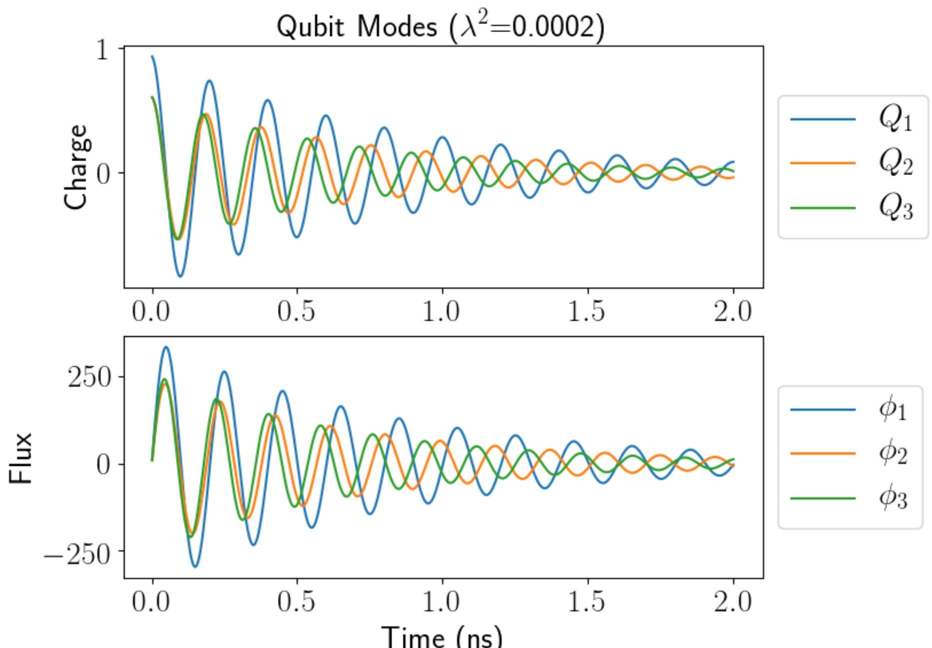
We have simulated the Markovian dynamics of the Brune circuit for different choices of circuit parameters

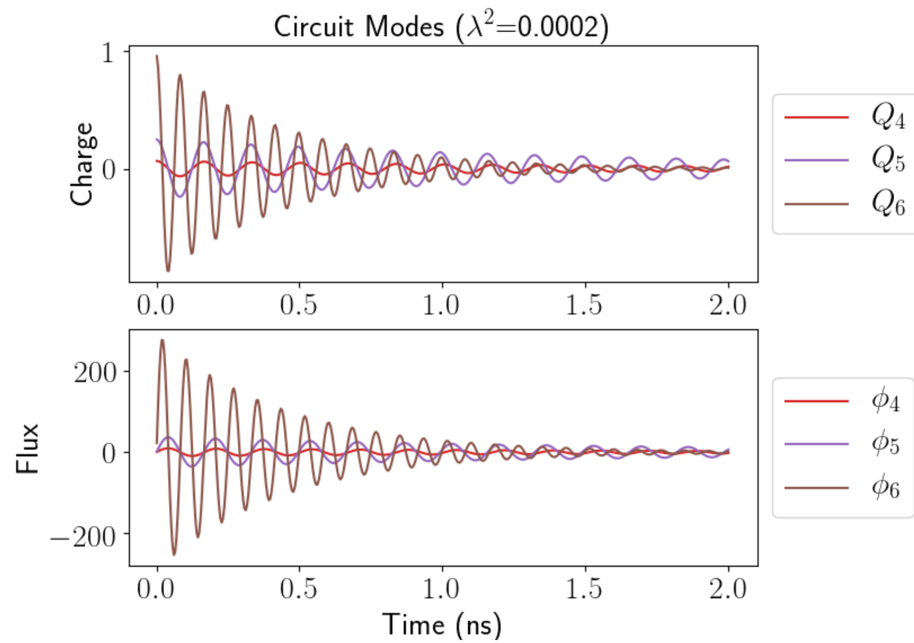
Cortes et al. J.Chem.Phys. **82** (6) 1985

Uncoupled system operators

$$\mathcal{C}_0^{-1} = \begin{pmatrix} C_{J_1}^{-1} & & & & & \\ & C_{J_2}^{-1} & & & & \\ & & C_{J_3}^{-1} & & & \\ & & & C_1^{-1} & & \\ & & & & C_2^{-1} & \\ & & & & & C_3^{-1} \end{pmatrix} \quad \mathbf{M}_0 = \begin{pmatrix} L_{J_1}^{-1} & & & & & \\ & L_{J_2}^{-1} & & & & \\ & & L_{J_3}^{-1} & & & \\ & & & L_1^{-1} & & \\ & & & & L_2^{-1} & \\ & & & & & L_3^{-1} \end{pmatrix}$$

Capacitance (fF)	C_{J_1}	C_{J_2}	C_{J_3}	C_1	C_2	C_3
	84.4	75.1	67.3	175.9	173.0	43.6
Inductance (nH)	L_{J_1}	L_{J_2}	L_{J_3}	L_1	L_2	L_3
	12.0	12.0	12.0	4.0	4.0	4.0
$\omega/2\pi$ (GHz)	ω_{J_1}	ω_{J_2}	ω_{J_3}	ω_1	ω_2	ω_3
	5.0	5.3	5.6	5.95	6.05	12.05





Conclusion / Main observation: Charges and fluxes evolve as damped harmonic oscillators, dissipate at different rates

Symmetric nearest neighbors cross-talk

$$\widetilde{\mathcal{C}}_0^{-1} = \begin{pmatrix} C_{J_1}^{-1} & C_{12}^{-1} & & & & \\ C_{21}^{-1} & C_{J_2}^{-1} & C_{23}^{-1} & & & \\ & C_{32}^{-1} & C_{J_3}^{-1} & & & \\ & & & C_1^{-1} & & \\ & & & & C_2^{-1} & \\ & & & & & C_3^{-1} \end{pmatrix} \quad \mathbf{M}_0 = \begin{pmatrix} L_{J_1}^{-1} & & & & & \\ & L_{J_2}^{-1} & & & & \\ & & L_{J_3}^{-1} & & & \\ & & & L_1^{-1} & & \\ & & & & L_2^{-1} & \\ & & & & & L_3^{-1} \end{pmatrix}$$

Coupling Capacitance (fF)	C_{21}	C_{21}	C_{32}	C_{23}		
	318.5	318.5	284.1	284.1		
Capacitance (fF)	C_{J_1}	C_{J_2}	C_{J_3}	C_1	C_2	C_3
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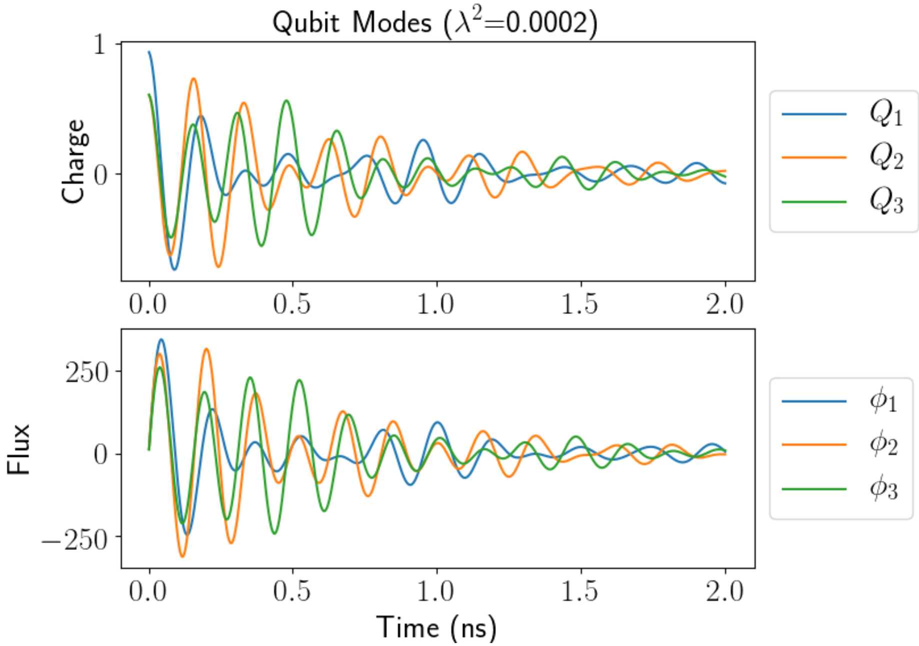
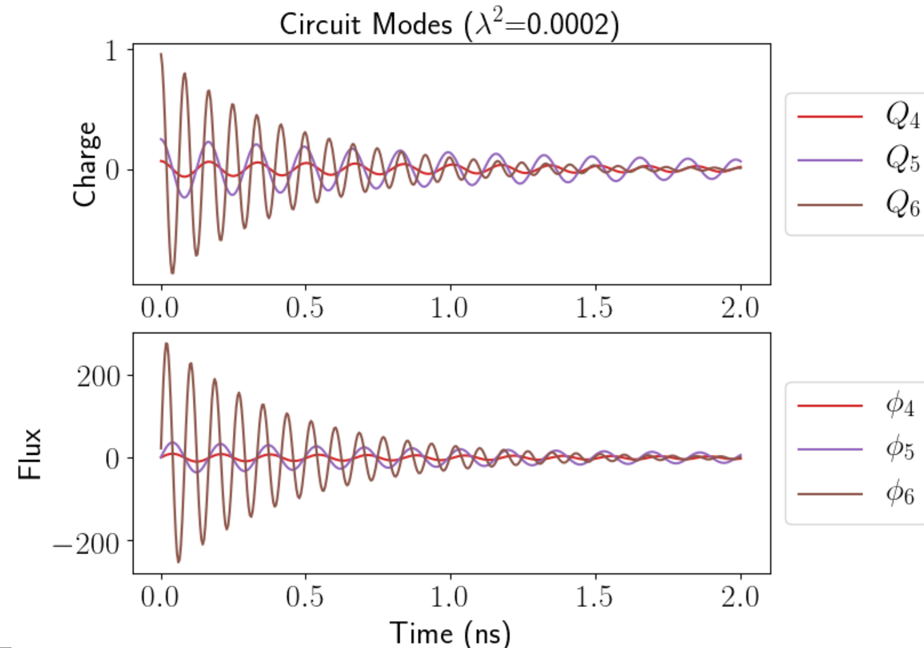


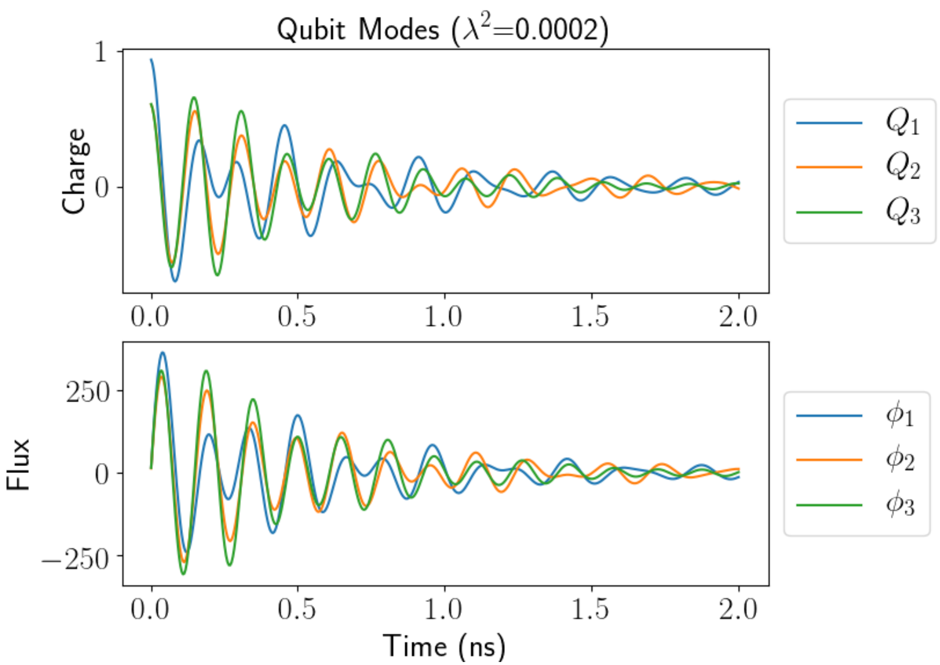


Conclusion / Main observation: Coupling introduces beat frequencies, energy in the uncoupled case is transferred from the JJ's to the circuit

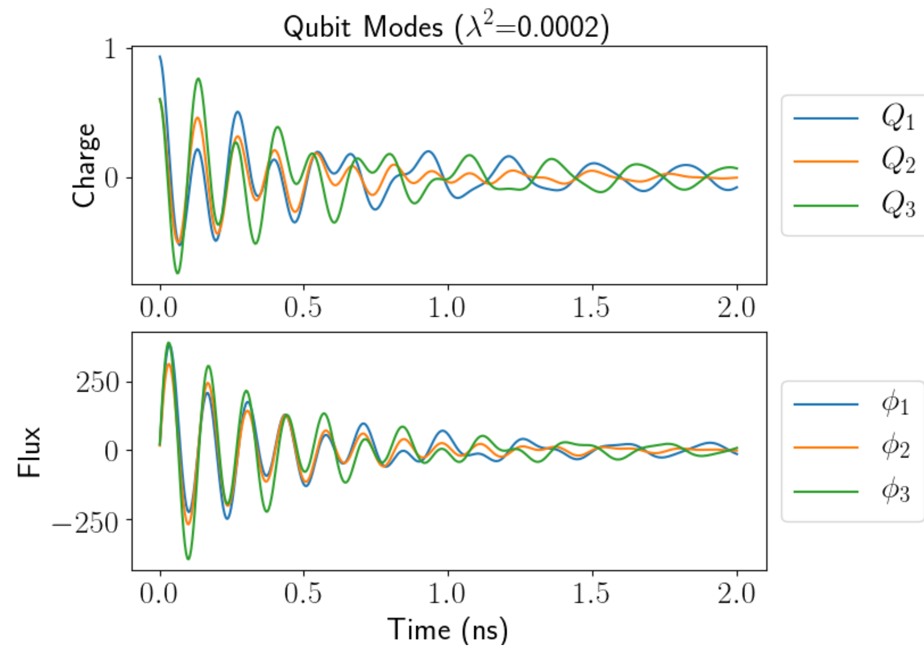
All-to-all cross-talk

$$\widetilde{\mathcal{C}}_0^{-1} = \begin{pmatrix} C_{J_1}^{-1} & C_{12}^{-1} & C_{13}^{-1} & & & \\ C_{21}^{-1} & C_{J_2}^{-1} & C_{23}^{-1} & & & \\ C_{31}^{-1} & C_{32}^{-1} & C_{J_3}^{-1} & & & \\ & & & C_1^{-1} & & \\ & & & & C_2^{-1} & \\ & & & & & C_3^{-1} \end{pmatrix} \quad \mathbf{M}_0 = \begin{pmatrix} L_{J_1}^{-1} & & & & & \\ & L_{J_2}^{-1} & & & & \\ & & L_{J_3}^{-1} & & & \\ & & & L_1^{-1} & & \\ & & & & L_2^{-1} & \\ & & & & & L_3^{-1} \end{pmatrix}$$

Symmetric Coupling



Asymmetric Coupling



Conclusion / Main observation: Coupling introduces beat frequencies, energy in the uncoupled case is transferred from the JJ's to the circuit

Summary and future work

Summary

- Derived quantum Langevin equations for multiport circuit
- Developed and exercised software to numerically solve for Markovian dynamics of charge and flux

Future Work

- Simulate non-Markovian dynamics of charge and flux
- Derive higher order moments for c-number Langevin equations[4]
- Calculate fluctuations about the mean values
- Calculate two-time correlation functions to study correlated processes in circuits
- Apply the method to results of FEM model of physical devices
- Include experimental data in the calculations
- Exercise the approach on multiple bath models

[4] W. Louisell, Quantum statistical properties of radiation, Wiley Series in Pure and Applied Optics Series (John Wiley & Sons Canada, Limited, 1973).

Acknowledgements

- This work was funded by the LLNL Laboratory Directed Research and Development (LDRD) program, project number 18-ERD-039.

Additional / Old Slides

Asymmetric tridiagonal coupling

$$\tilde{\mathcal{C}}_0^{-1} = \begin{pmatrix} C_{J_1}^{-1} & C_{12}^{-1} & & & & \\ C_{21}^{-1} & C_{J_2}^{-1} & C_{23}^{-1} & & & \\ & C_{32}^{-1} & C_{J_3}^{-1} & C_{34}^{-1} & & \\ & & C_{43}^{-1} & C_1^{-1} & C_{45}^{-1} & \\ & & & C_{54}^{-1} & C_2^{-1} & C_{56}^{-1} \\ & & & & C_{65}^{-1} & C_3^{-1} \end{pmatrix} \quad \mathbf{M}_0 = \begin{pmatrix} L_{J_1}^{-1} & & & & & \\ & L_{J_2}^{-1} & & & & \\ & & L_{J_3}^{-1} & & & \\ & & & L_1^{-1} & & \\ & & & & L_2^{-1} & \\ & & & & & L_3^{-1} \end{pmatrix}$$

Coupling Capacitance (fF)	C_{21}	C_{32}	C_{43}	C_{54}	C_{65}	
	159.3	142.2	217.6	175.9	58.6	
Capacitance (fF)	C_{J_1}	C_{J_2}	C_{J_3}	C_1	C_2	C_3
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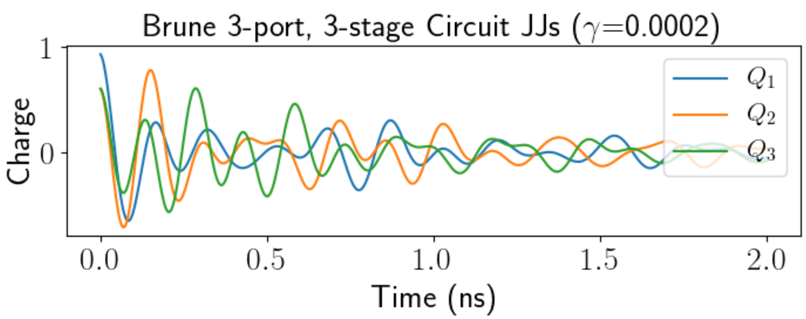
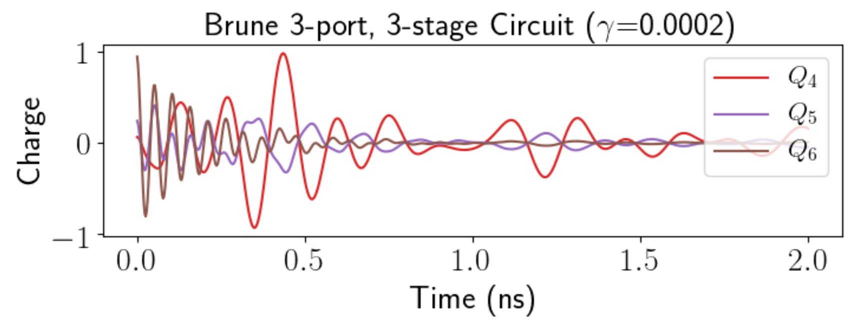
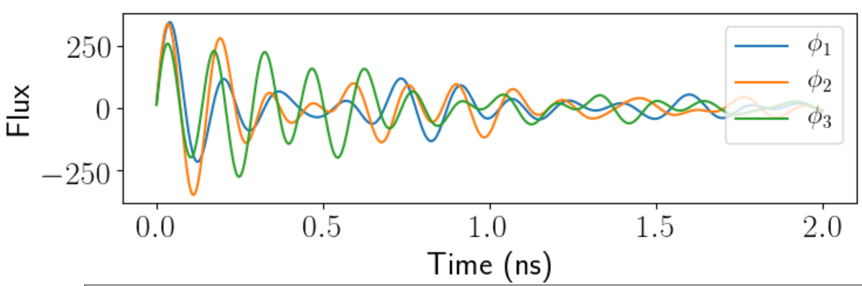
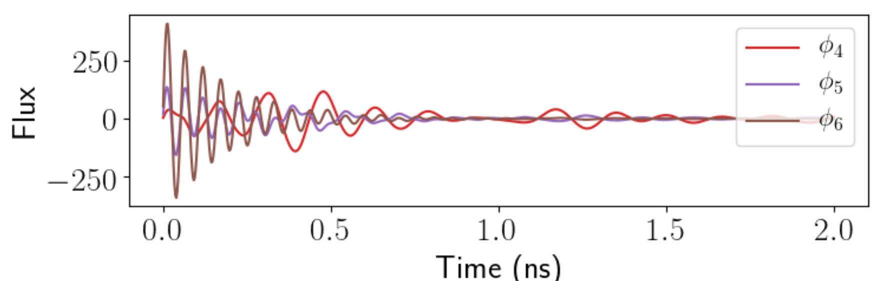
$\omega/2\pi$ (GHz)	ω_{J_1}	ω_{J_2}	ω_{J_3}	ω_1	ω_2	ω_3
	5.0	5.3	5.6	6.0	12.0	18.0









Conclusion / Main observation: Asymmetries in capacitance matrix modify spectral content of circuit and qubit variables

Analytic memory kernel – Ornstein-Uhlenbeck random process, equations of motion

$$\dot{\phi}_k(t) = y_k(t)$$

$$\dot{y}_k(t) = -\frac{2}{2}\left(\sum_i \mathcal{C}^{-1}_{0ik} M_{0ii}\phi_i(t) + \sum_j \mathcal{C}^{-1}_{0kj} M_{0jj}\phi_j(t)\right)$$

$$+\frac{\lambda^2}{2}\sum_\alpha \frac{c_\alpha^2}{m_\alpha \omega_\alpha^2}\left(\sum_i \mathcal{C}^{-1}_{0ik}\phi_i + \sum_j \mathcal{C}^{-1}_{0kj}\phi_j\right)$$

$$+\frac{1}{2}\frac{\Phi_0}{2\pi}\left(\sum_i \mathcal{C}^{-1}_{0ik} L_{J_{i\leq N_{\mathrm{JJ}}}} \sin\left(\frac{2\pi}{\Phi_0}\phi_i(t)\right)\right.$$

$$\left.+\sum_j \mathcal{C}^{-1}_{0kj} L_{J_{j\leq N_{\mathrm{JJ}}}} \sin\left(\frac{2\pi}{\Phi_0}\phi_j(t)\right)\right)$$

$$+\xi_k(t) + w_k(t)$$

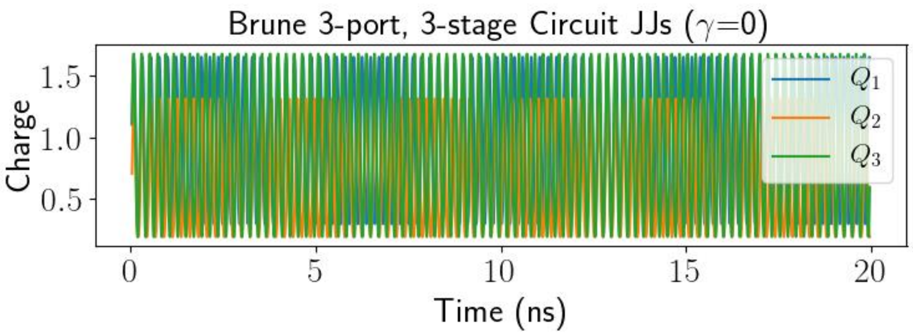
$$\dot{w}_k(t) = -\gamma\frac{1}{2}\lambda^2\left(\sum_i \mathcal{C}^{-1}_{0ik} w_i(t) + \sum_j \mathcal{C}^{-1}_{0kj} w_j(t)\right) - \mathscr{K}(0) y_k(t)$$

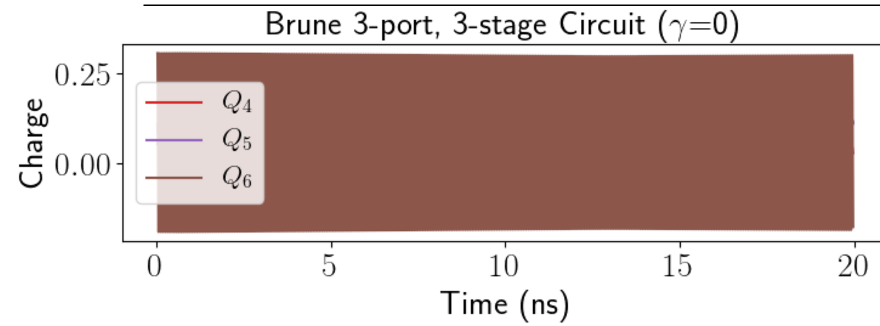
$$\dot{\xi}_k(t) = -\gamma\left[\xi_k(t) - \sqrt{2T\eta}\,\zeta(t)\right]$$

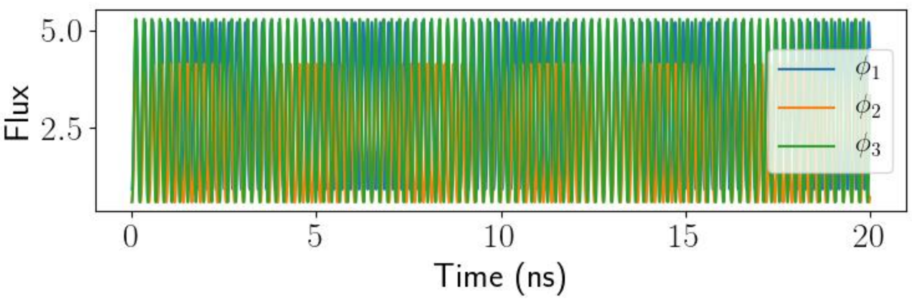
Analytic memory kernel – Ornstein-Uhlenbeck random process, dissipationless limit

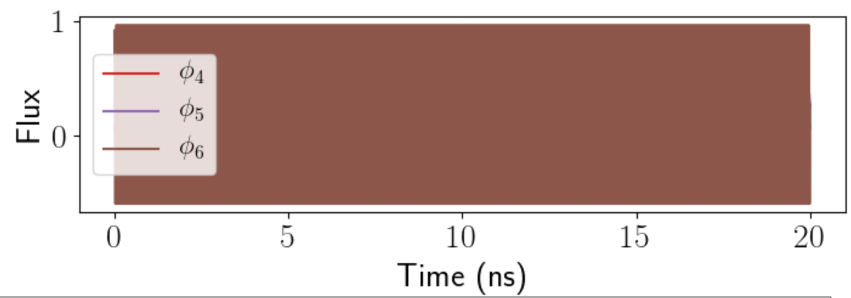
$$\mathcal{C}_0^{-1} = \begin{pmatrix} C_{J_1}^{-1} & & & & & \\ & C_{J_2}^{-1} & & & & \\ & & C_{J_3}^{-1} & & & \\ & & & C_1^{-1} & & \\ & & & & C_2^{-1} & \\ & & & & & C_3^{-1} \end{pmatrix} \mathbf{M}_0 = \begin{pmatrix} L_{J_1}^{-1} & & & & & \\ & L_{J_2}^{-1} & & & & \\ & & L_{J_3}^{-1} & & & \\ & & & L_1^{-1} & & \\ & & & & L_2^{-1} & \\ & & & & & L_3^{-1} \end{pmatrix}$$

Capacitance (fF)	C_{J_1}	C_{J_2}	C_{J_3}	C_1	C_2	C_3
	84.4	75.1	67.3	175.9	44.0	19.5
Inductance (nH)	L_{J_1}	L_{J_2}	L_{J_3}	L_1	L_2	L_3
	12.0	12.0	12.0	4.0	4.0	4.0
$\omega/2\pi$ (GHz)	ω_{J_1}	ω_{J_2}	ω_{J_3}	ω_1	ω_2	ω_3
	5.0	5.3	5.6	6.0	12.0	18.0







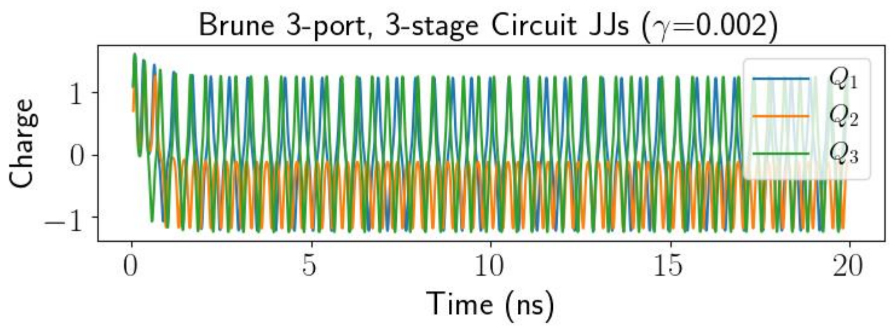


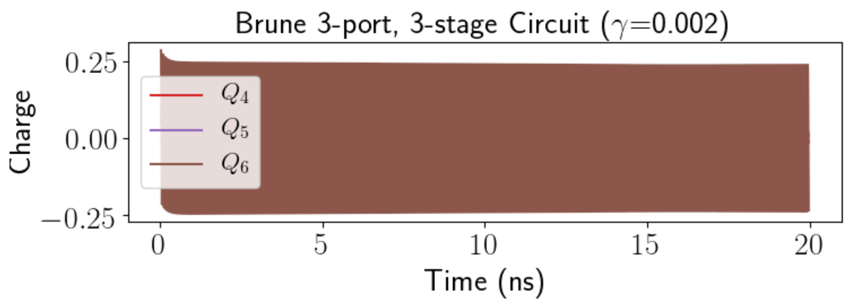
Conclusion / Main observation: In this limit there is no dissipation

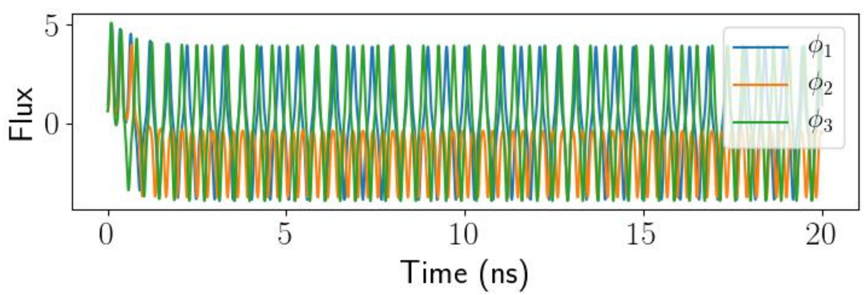
Analytic memory kernel – Ornstein-Uhlenbeck random process, non-Markovian limit

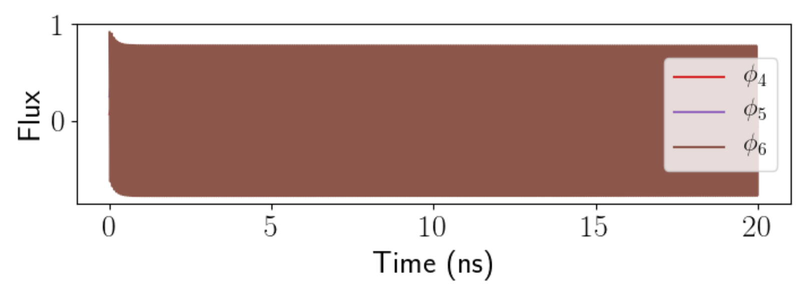
$$\mathcal{C}_0^{-1} = \begin{pmatrix} C_{J_1}^{-1} & & & & & \\ & C_{J_2}^{-1} & & & & \\ & & C_{J_3}^{-1} & & & \\ & & & C_1^{-1} & & \\ & & & & C_2^{-1} & \\ & & & & & C_3^{-1} \end{pmatrix} \mathbf{M}_0 = \begin{pmatrix} L_{J_1}^{-1} & & & & & \\ & L_{J_2}^{-1} & & & & \\ & & L_{J_3}^{-1} & & & \\ & & & L_1^{-1} & & \\ & & & & L_2^{-1} & \\ & & & & & L_3^{-1} \end{pmatrix}$$

Capacitance (fF)	C_{J_1}	C_{J_2}	C_{J_3}	C_1	C_2	C_3
	84.4	75.1	67.3	175.9	44.0	19.5
Inductance (nH)	L_{J_1}	L_{J_2}	L_{J_3}	L_1	L_2	L_3
	12.0	12.0	12.0	4.0	4.0	4.0
$\omega/2\pi$ (GHz)	ω_{J_1}	ω_{J_2}	ω_{J_3}	ω_1	ω_2	ω_3
	5.0	5.3	5.6	6.0	12.0	18.0







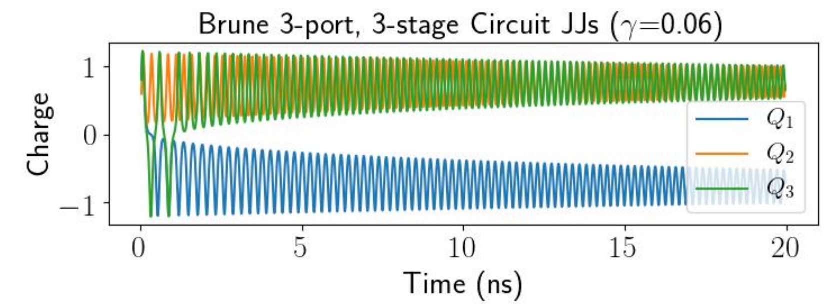


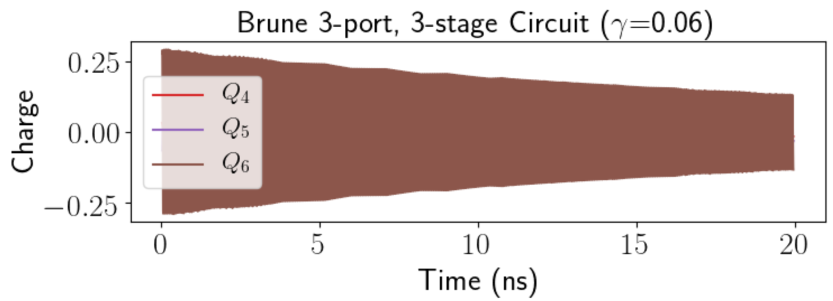
Conclusion / Main observation: Circuit DOF's dissipate at a slower rate, JJ DOF's exhibit more non-linearity

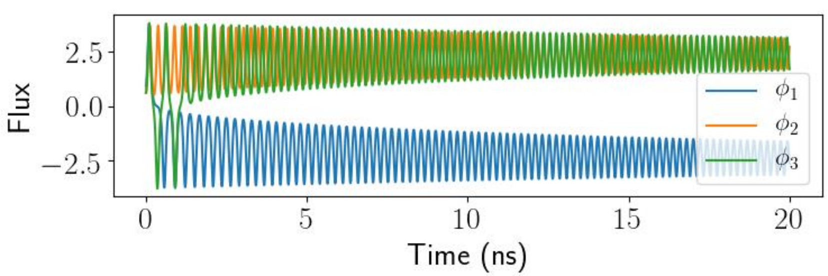
Analytic memory kernel – Ornstein-Uhlenbeck random process, non-Markovian

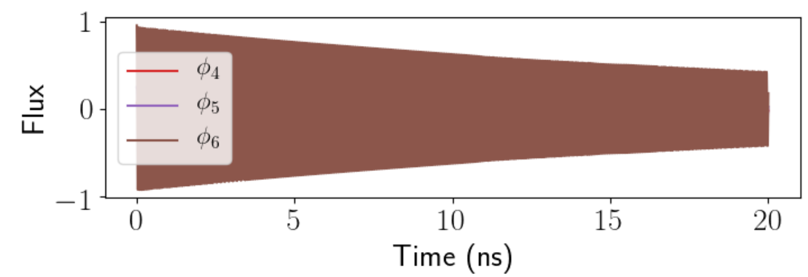
$$\mathcal{C}_0^{-1} = \begin{pmatrix} C_{J_1}^{-1} & & & & & \\ & C_{J_2}^{-1} & & & & \\ & & C_{J_3}^{-1} & & & \\ & & & C_1^{-1} & & \\ & & & & C_2^{-1} & \\ & & & & & C_3^{-1} \end{pmatrix} \mathbf{M}_0 = \begin{pmatrix} L_{J_1}^{-1} & & & & & \\ & L_{J_2}^{-1} & & & & \\ & & L_{J_3}^{-1} & & & \\ & & & L_1^{-1} & & \\ & & & & L_2^{-1} & \\ & & & & & L_3^{-1} \end{pmatrix}$$

Capacitance (fF)	C_{J_1}	C_{J_2}	C_{J_3}	C_1	C_2	C_3
	84.4	75.1	67.3	175.9	44.0	19.5
Inductance (nH)	L_{J_1}	L_{J_2}	L_{J_3}	L_1	L_2	L_3
	12.0	12.0	12.0	4.0	4.0	4.0
$\omega/2\pi$ (GHz)	ω_{J_1}	ω_{J_2}	ω_{J_3}	ω_1	ω_2	ω_3
	5.0	5.3	5.6	6.0	12.0	18.0







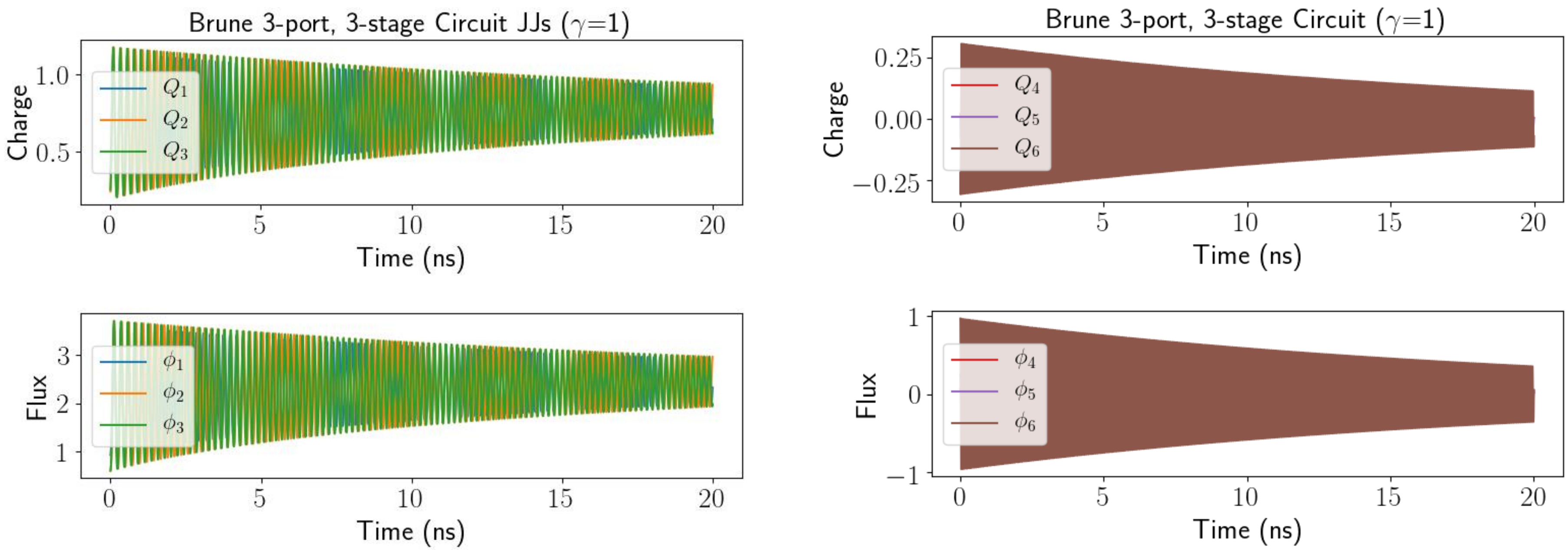


Conclusion / Main observation: The circuit DOF's dissipate as harmonic oscillators and the JJ DOF's transition to more damped behavior

Analytic memory kernel – Ornstein-Uhlenbeck random process, Markovian limit

$$\mathcal{C}_0^{-1} = \begin{pmatrix} C_{J_1}^{-1} & & & & & \\ & C_{J_2}^{-1} & & & & \\ & & C_{J_3}^{-1} & & & \\ & & & C_1^{-1} & & \\ & & & & C_2^{-1} & \\ & & & & & C_3^{-1} \end{pmatrix} \mathbf{M}_0 = \begin{pmatrix} L_{J_1}^{-1} & & & & & \\ & L_{J_2}^{-1} & & & & \\ & & L_{J_3}^{-1} & & & \\ & & & L_1^{-1} & & \\ & & & & L_2^{-1} & \\ & & & & & L_3^{-1} \end{pmatrix}$$

Capacitance (fF)	C_{J_1}	C_{J_2}	C_{J_3}	C_1	C_2	C_3
	84.4	75.1	67.3	175.9	44.0	19.5
Inductance (nH)	L_{J_1}	L_{J_2}	L_{J_3}	L_1	L_2	L_3
	12.0	12.0	12.0	4.0	4.0	4.0
$\omega/2\pi$ (GHz)	ω_{J_1}	ω_{J_2}	ω_{J_3}	ω_1	ω_2	ω_3
	5.0	5.3	5.6	6.0	12.0	18.0



Conclusion / Main observation: Both the circuit and JJ DOF's dissipate, although at a slower rate, in a similar fashion as in the Markovian case

Classical Langevin equation

The classical Langevin equation for the charges in the circuit can be written following the convention in [14]

$$\dot{Q}_k + \frac{\partial H_S^{(m)}}{\partial \Phi} + \lambda^2 \int_0^t K\left(t-\tau\right) \dot{\phi}_k\left(\tau\right) d\tau = \lambda F(t)$$

$$H_S^{(m)} = H_S - \lambda^2 \sum_\alpha \frac{c_\alpha^2}{2 m_\alpha \omega_\alpha^2} \phi_k^2(t) + \frac{\Phi_0}{2\pi} \sum_{j \leq N_{\mathrm{JJ}}} L_{J_j}^{-1} \sin\left(\phi_j \frac{2\pi}{\Phi_0}\right) \delta_{jk}$$

$$K(t-\tau) = \sum_\alpha \frac{c_\alpha^2}{m_\alpha \omega_\alpha^2} \cos(\omega_\alpha (t-\tau))$$

$$F(t) = \sum_\alpha c_\alpha \left\{ \left[x_\alpha(0) - \frac{2\pi}{\Phi_0} \sum_{ij} \bar{m}_{ij} \frac{c_\alpha}{m_\alpha \omega_\alpha^2} \phi_j(0) \right] \cos(\omega_\alpha t) + \frac{p_\alpha(0)}{m_\alpha \omega_\alpha} \sin(\omega_\alpha t) \right\}$$

$$\lambda = \frac{2\pi}{\Phi_0} \sum_i \bar{m}_{ik}$$

[14] Cortes et al. J.Chem.Phys. **82** (6) 1985

Classical fluctuation dissipation relation

To calculate the classical fluctuation dissipation relation, one needs to compute the two time correlation function

$$\langle F(t)F(\tau)\rangle = \sum_{\alpha} \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_{\alpha}(0)dp_{\alpha}(0)e^{-\beta H_B^{(m)}} F_{\alpha}(t)F_{\alpha}(\tau)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_{\alpha}(0)dp_{\alpha}(0)e^{-\beta H_B^{(m)}}}$$

where the modified bath Hamiltonian has coordinates shifted from the original bath coordinates[14]

$$H_B^{(m)} = \sum_{\alpha} \frac{p_{\alpha}^2}{2m_{\alpha}} + \frac{1}{2}m_{\alpha}\omega_{\alpha}^2 \left[x_{\alpha} - \frac{2\pi}{\Phi_0}\sum_{ij}\bar{m}_{ij}\frac{c_{\alpha}}{m_{\alpha}\omega_{\alpha}}\phi_j\right]^2$$

[14] Cortes et al. J.Chem.Phys. **82** (6) 1985

Classical fluctuation dissipation relation

Evaluating the integrals we arrive at an expression for the correlation function

$$\langle F(t)F(\tau)\rangle = \frac{1}{\beta}\sum_{\alpha}\frac{c_\alpha^2}{m_\alpha\omega_\alpha^2}\cos(\omega_\alpha(t-\tau))$$

We can identify the memory kernel and arrive at the fluctuation dissipation relation for the Brune multiport circuit coupled to a classical harmonic bath

$$\langle F(t)F(\tau)\rangle = k_B T \; K(t-\tau)$$

where T is the temperature of the bath and k_B is Boltzmann's constant

Quantum derivation

Starting from the classical Hamiltonians, we replace the variable with vectors of operators, expressed as sums over the elements of the coupling matrices[13,14]

$$\hat{H}_S = \frac{1}{2}\sum_{ij} \mathcal{C}_{0ij}^{-1}\hat{Q}_i\hat{Q}_j + \hat{U}(\hat{\Phi})$$

The second term in the system Hamiltonian is the potential. It may contain non-linear terms describing Josephson junctions in the circuit as

$$\hat{U}_J(\hat{\Phi}) = -\left(\frac{\Phi_0}{2\pi}\right)^2 \sum_{j\leq N_{\mathrm{JJ}}} L_{J_j}^{-1}\cos\left(\frac{2\pi}{\Phi_0}\hat{\phi}_j\right)$$

We expand the Josephson potential to second order to simplify the analysis

$$\hat{U}(\hat{\Phi}) \approx \frac{1}{2}\sum_{ij} M_{0ij}\hat{\phi}_i\hat{\phi}_j - \left(\frac{\Phi_0}{2\pi}\right)^2 \sum_{j\leq N_{\mathrm{JJ}}} L_{J_j}^{-1}\left(1-\left(\frac{2\pi}{\Phi_0}\right)^2\hat{\phi}_j^2/2\right)$$

[13] Solgun et al. PRB, **90** 134504 (2014)
[14] Solgun, PhD Thesis, RWTH, (2015)

Quantum derivation

Recall in the classical system that the charge and flux are canonically conjugate variables. We express this relationship quantum mechanically with the commutation relation[1]

$$\left[\hat{\phi}_i,\ \hat{Q}_j\right] = i\hbar\delta_{ij}$$

The equations of motion in the Heisenberg picture rely on these commutation relations along with those describing the bath degrees of freedom. For a harmonic bath we have

$$\hat{H}_B = \sum_\alpha \frac{\hat{p}_\alpha}{2m_\alpha} + \frac{1}{2} m_\alpha \omega_\alpha^2 \hat{x}_\alpha^2 = \sum_\alpha \hbar\omega_\alpha \left(\hat{b}_\alpha^\dagger \hat{b}_\alpha + 1/2\right)$$

with bosonic creation and annihilation operators given by

$$\hat{b}_\alpha^\dagger \left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle, \qquad \hat{b}_\alpha \left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$$

$$\left[\hat{b}_\alpha,\ \hat{b}_\beta^\dagger\right] = \delta_{\alpha\beta}, \qquad \left[\hat{b}_\alpha,\ \hat{b}_\beta\right] = 0, \qquad \left[\hat{b}_\alpha^\dagger,\ \hat{b}_\beta^\dagger\right] = 0$$

[1] Burkard et al. PRB, **69** 064503 (2004)

Quantum derivation

The interaction Hamiltonian between the system and the bath degrees of freedom follows a similar bilinear model as in the classical case

$$\hat{H}_{SB} = \frac{2\pi}{\Phi_0} \sum_{\alpha i j} \bar{m}_{ij} c_\alpha X_\alpha^{\mathrm{ZPF}} \hat{\phi}_j \left(\hat{b}_\alpha + \hat{b}_\alpha^\dagger \right)$$

$$X_\alpha^{\mathrm{ZPF}} = \sqrt{\frac{\hbar}{2 m_\alpha \omega_\alpha}}, \qquad \hat{\phi}_i = \sqrt{\frac{\hbar}{2\omega_i C_i}} \left(\hat{a}_i + \hat{a}_i^\dagger \right),$$

$$\implies \left[\hat{\phi}_i, \hat{b}_\alpha \right] = 0, \quad \left[\hat{\phi}_i, \hat{b}_\alpha^\dagger \right] = 0, \quad \left[\hat{Q}_i, \hat{b}_\alpha \right] = 0, \quad \left[\hat{Q}_i, \hat{b}_\alpha^\dagger \right] = 0$$

Using these relations, we solve for the quantum Langevin equation for each charge in the circuit[14]

[14] Cortes et al. J.Chem.Phys. **82** (6) 1985

Quantum Langevin equation

The quantum Langevin equation for a charge on the *k-th* capacitor in the k-th multiport Brune stage is given in [14] by

$$\frac{d\hat{Q}_k}{dt} + \frac{i}{\hbar}\left[\hat{H}_S^{(m)},\ \hat{Q}_k\right] + \lambda^2 \int_0^t \mathscr{K}\left(t-\tau\right)\dot{\phi}_k\left(\tau\right)d\tau = \lambda\hat{\mathscr{F}}(t)$$

where,

$$\hat{H}_S^{(m)} = \hat{H}_S - \lambda^2 \sum_\alpha \frac{c_\alpha^2}{2m_\alpha\omega_\alpha^2}\hat{\phi}_k^2 - \sum_{j\le N_{\mathrm{JJ}}} L_{J_j}^{-1}\delta_{kj}\hat{\phi}_j^2$$

$$\mathscr{K}\left(t-\tau\right) = \sum_\alpha \frac{c_\alpha^2}{m_\alpha\omega_\alpha^2}\cos\left(\omega_\alpha\left(t-\tau\right)\right)$$

$$\hat{\mathscr{F}}(t) = \sum_\alpha c_\alpha X_\alpha^{\mathrm{ZPF}}\left\{\left[\hat{b}_\alpha(0) - \lambda\frac{c_\alpha}{\hbar\omega_\alpha}X_\alpha^{\mathrm{ZPF}}\hat{\phi}_k(0)\right]e^{-i\omega_\alpha t} + \mathrm{h.c.}\right\}$$

$$\lambda = \frac{2\pi}{\Phi_0}\sum_i \bar{m}_{ik}$$

[14] Cortes et al. J.Chem.Phys. **82** (6) 1985

Quantum Fluctuation Dissipation Relation

We are now interested in calculated the two time correlation function of the fluctuating force to identify the fluctuation dissipation relation. First, we will define a modified bath with operators constructed by Bogoliubov transformation[14]

$$\hat{H}_B^{(m)} = \sum_\alpha \hbar w_\alpha \hat{B}_\alpha^\dagger \hat{B}_\alpha$$

$$\hat{B}_\alpha = \hat{b}(0)_\alpha - \lambda \frac{c_\alpha}{\hbar\omega_\alpha}\hat{\phi}_k(0)$$

$$\left[\hat{B}_\alpha, \hat{B}_\beta^\dagger\right] = \delta_{\alpha\beta}, \quad \left[\hat{B}_\alpha^{(\dagger)}, \hat{B}_\alpha^{(\dagger)}\right] = 0$$

$$\hat{\mathscr{F}}(t) = \sum_\alpha c_\alpha X_\alpha^{\mathrm{ZPF}} \left(\hat{B}_\alpha e^{-i\omega_\alpha t} + \hat{B}_\alpha^\dagger e^{i\omega_\alpha t}\right)$$

[14] Cortes et al. J.Chem.Phys. **82** (6) 1985

Quantum Fluctuation Dissipation Relation

To compute the expectation operators, we follow Cortes et al. and take the density matrix of the bath to describe our Gaussian random operator as[14]

$$\hat{\rho}_B^{(m)} = \frac{e^{-\hat{H}_B^{(m)}/k_B T}}{\mathrm{Tr}\; e^{-\hat{H}_B^{(m)}/k_B T}} = \frac{e^{-\hat{H}_B^{(m)}/k_B T}}{Z}$$

with the expectation values of the shifted bath operators are then given by

$$\left\langle \hat{B}_\alpha^\dagger \hat{B}_\alpha \right\rangle = \mathrm{Tr}\; \hat{\rho}_B^{(m)} \hat{B}_\alpha^\dagger \hat{B}_\alpha = \frac{1}{e^{\hbar\omega_\alpha/k_B T}} = n_\alpha$$

and the two time correlation function reads

$$\left\langle \hat{\mathscr{F}}(t)\hat{\mathscr{F}}(\tau) \right\rangle = \sum_\alpha c_\alpha^2 \frac{\hbar}{2m_\alpha \omega_\alpha} \left[(2n_\alpha + 1)\cos(\omega_\alpha(t-\tau)) - i\sin(\omega_\alpha(t-\tau)) \right]$$

[14] Cortes et al. J.Chem.Phys. **82** (6) 1985

Quantum Fluctuation Dissipation Relation

To relate the two time correlation function to the memory kernel, we calculate the symmetrized correlation function[14]

$$\left\langle \hat{\mathscr{F}}(t)\hat{\mathscr{F}}(\tau)\right\rangle + \left\langle \hat{\mathscr{F}}(\tau)\hat{\mathscr{F}}(t)\right\rangle = \sum_\alpha c_\alpha^2 \frac{\hbar}{m_\alpha \omega_\alpha}(2n_\alpha + 1)\cos(\omega_\alpha(t-\tau))$$

$$= \sum_\alpha c_\alpha^2 \frac{\hbar}{m_\alpha \omega_\alpha} \cos(\omega_\alpha(t-\tau))\coth\left(\frac{\hbar w_\alpha}{2k_B T}\right)$$

$$= \sum_\alpha \Phi_\alpha(t-\tau)$$

The fluctuation dissipation relation then follows[14]

$$\mathscr{K}(t-\tau) = \sum_\alpha \Phi_\alpha(t-\tau)\frac{\tanh\left(\hbar w_\alpha/2k_B T\right)}{\hbar \omega_\alpha}$$

[14] Cortes et al. J.Chem.Phys. **82** (6) 1985

Coupled quantum equations of motion

The charge and flux operators' time evolution are coupled through the derivative of the flux in the Langevin equation. This leads to the following coupled ordinary differential equations (ODE's)

$$\frac{d\hat{Q}_k}{dt} + \frac{i}{\hbar}\left[\hat{H}_S^{(m)},\ \hat{Q}_k\right] + \lambda^2 \int_0^t \mathscr{K}\left(t-\tau\right)\dot{\phi}_k\left(\tau\right)d\tau = \lambda\hat{\mathscr{F}}(t)$$

$$\frac{d\hat{\phi}_k}{dt} = \frac{1}{2}\left(\sum_i \mathcal{C}_{0ik}^{-1}\hat{Q}_i + \sum_j \mathcal{C}_{0kj}^{-1}\hat{Q}_j\right)$$

We note that the right hand side of the first equation is zero centered, e.g. it has zero mean. This simplifies the calculation of the mean value equations, or the first moment equations.

Coupled quantum equations of motion

The mean equations of charge and flux operators

$$\frac{d}{dt}\left\langle \hat{Q}_k \right\rangle + \frac{1}{2}\left(\sum_i M_{0ik}\left\langle \hat{\phi}_i \right\rangle + \sum_j M_{0kj}\left\langle \hat{\phi}_j \right\rangle \right) - \sum_{j \le N_{\mathrm{JJ}}} L_{J_j}^{-1}\delta_{kj}\left\langle \hat{\phi}_j \right\rangle$$

$$-\lambda^2 \sum_\alpha \frac{c_\alpha^2}{m_\alpha \omega_\alpha^2}\left\langle \hat{\phi}_k \right\rangle + \lambda^2 \int_0^t \mathscr{K}(t-\tau)\frac{d}{d\tau}\left\langle \hat{\phi}_k \right\rangle d\tau = 0$$

$$\frac{d}{dt}\left\langle \hat{\phi}_k \right\rangle - \frac{1}{2}\left(\sum_i \mathcal{C}_{0ik}^{-1}\left\langle \hat{Q}_i \right\rangle + \sum_j \mathcal{C}_{0kj}^{-1}\left\langle \hat{Q}_j \right\rangle \right) = 0$$

The equations above give the full non-Markovian dynamics of the system. We will look at the Markovian limit of these equations, where the memory kernel behaves as a delta function or the fluctuation-dissipation relation describes a white noise source.

non-Markovian Equations

We will now evaluate the sums over the bath coordinates as integrals over a continuum of modes, starting with the memory kernel. A DeBye density of states, $g(\omega)$, is chosen to illustrate partially non-trivial integration of the memory kernel.

$$\sum_{\alpha}(\cdot) \to \int g(\omega)(\cdot)\, d\omega, \;\; \omega_\alpha \to \omega$$

$$\int_{-\infty}^{\infty} \frac{c_\alpha^2}{m_\alpha \omega^2} g(\omega) \cos(\omega(t-\tau))\, d\omega = \int_0^{\omega_D} \frac{c_\alpha^2}{m_\alpha \omega^2} \left(\frac{V}{c^3}\omega^2\right) \cos(\omega(t-\tau))\, d\omega$$

$$= \frac{V c_\alpha^2}{m_\alpha c^3} \omega_D \frac{\sin(\omega_D(t-\tau))}{\omega_D(t-\tau)}$$

$$= \frac{V c_\alpha^2}{m_\alpha c^3} \omega_D \operatorname{sinc}(\omega_D(t-\tau))$$

The time dependent result is a sinc function, a damped, oscillating function symmetric about $t=\tau$, whose limit is unity at $t=\tau$.

non-Markovian Equations

Evaluating the integral in the frequency shift with the same density of states and collecting all of the coefficients gives the mean charge equation

$$\frac{dQ_k}{dt} = -\frac{1}{2}\left(\sum_i M_{0ik}\phi_i + \sum_j M_{0kj}\phi_j\right) + \sum_{j \leq N_{\mathrm{JJ}}} L_{J_j}^{-1}\delta_{kj}\phi_j$$

$$+\widetilde{\lambda}^2 \left\{\omega_D \phi_k - \int_0^t \frac{\sin(\omega_D(t-\tau))}{t-\tau}\frac{d}{d\tau}(\phi_k)d\tau\right\}$$

$$\widetilde{\lambda} = \lambda\sqrt{\frac{V c_\alpha^2}{m_\alpha c^3}}$$

The new damping parameter is scaled by the coefficients by integrating over all of the bath modes. This scaled parameter is lumped together in the numerical implementation of the equations of motion.

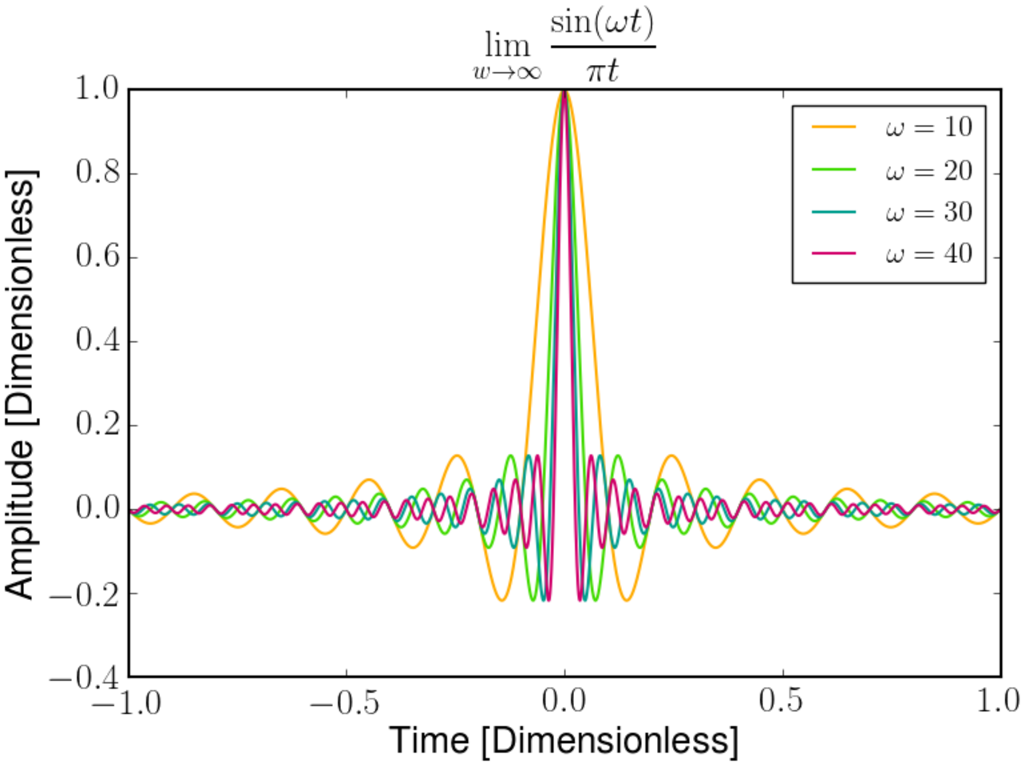
Markovian limit

We would like to show that the kernel of the integral on the right hand side of the previous equation for the charges acts like a delta function in some limit, e.g. the sifting theorem and normalization should hold[15]

$$\int_{-\infty}^{\infty} \frac{\sin(\omega_D(t-\tau))}{\pi(t-\tau)} d\tau = 1$$

$$\int_{-\infty}^{\infty} \frac{\sin(\omega_D(t-\tau))}{\pi(t-\tau)} f(\tau) d\tau \to f(t)$$

$$\lim_{\omega_D \to \infty} \frac{\sin(\omega_D(t-\tau))}{\pi(t-\tau)} = \delta(t-\tau)$$



[15] Arfken & Weber, *Essential Mathematical Methods for Physicists*

Markovian limit

Plugging in the modified memory kernel into the integral in the charge equation of motion and taking limits of integration out to infinity (ignoring the small errors) and applying the sifting theorem gives the Markovian equations of motion

$$\frac{dQ_k}{dt} = -\frac{1}{2}\left(\sum_i M_{0ik}\phi_i + \sum_j M_{0kj}\phi_j\right) + \sum_{j\leq N_{\mathrm{JJ}}} L_{J_j}^{-1}\delta_{kj}\phi_j + \widetilde{\lambda}^2\left(\omega_D\phi_k - \frac{d\phi_k}{dt}\right)$$

$$\frac{d\phi_k}{dt} = \frac{1}{2}\left(\sum_i \mathcal{C}_{0ik}^{-1}Q_i + \sum_j \mathcal{C}_{0kj}^{-1}Q_j\right)$$

Next, we solve this system of equations numerically using the GNU Scientific Library (libgsl) ODE solver tools in several limits for a model single stage multiport Brune circuit

Belevitch transformer structure